GEOMETRICAL NOTE ON VAN DER WAAL'S EQUATION*

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ABSTRACT. The object of the present paper is to study mathematically the (graphical) representation of van der Waal's equation in the (Euclidean) space of three dimensions. The investigations, conducted in this paper, centre round the geometry of the resulting graph (Ω), and takes account of the *Cremona* (or *birational*) transformations which convert Ω into a plane. Put in a nut-shell, the main results obtained are as follow :—

(*i*) that Ω is a unicursal quartic scroll and has a triple line at infinity;

(*ii*) that the line of striction of Ω is a unicursal quartic curve;

(*iii*) that the Hessian of Ω is a degenerate surface of the eighth degree, consisting of eight coincident planes;

(*iv*) that *every* polar quadric of Ω is a hyperbolic paraboloid;

(*v*) that the locus of a point, whose polar quadric is a pair of parallel planes, is virtually a plane;

(*vi*) that the '*critical point*' P (of Ω),—defined in the first instance as the point whose (Cartesian) co-ordinates are respectively the *critical pressure, critical volume* and *critical temperature*—is geometrically designable as the uniquely determinate point, having one of its inflexional tangents parallel to the axis of volume;

(*vii*) that the mean curvature of Ω vanishes at P; and,

(*viii*) that Ω has no curve of *zero* Gaussian curvature, although it has a curve of *zero* mean curvature.

INTRODUCTION

In the present paper I have discussed the geometrical representation of van der Waal's classical equation :

$$\left(p+\frac{a}{v^2}\right)(v-b)=\mathrm{RT}$$

in the (parabolic) space of three dimensions. No attempt has been made to enter into the merits of the underlying physical hypothesis. Rather the subject has been developed from a purely mathematical standpoint, and its interest lies mainly in the geometrical characterisation of the graph in question (Ω).

As a matter of convenience the subject has been sub-divided into four sections. Section I takes account of the line of striction of the quartic Ω, and

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an associated group of *birational* (or *Cremona*) transformations—very often contracted as *C.T.*'s. Section II deals with the Hessian and the system of polar quadrics (degenerate and non-degenerate) of the surface Ω. Section III treats principally of certain cardinal properties of the '*critical point*,' defined initially as the point (on Ω), whose Cartesian co-ordinates are respectively equal to the *critical pressure*, *critical volume* and *critical temperature*. Lastly Sec. IV disposes of certain organic curves of Ω, *e.g.*, curves of constant mean curvature or of constant specific curvature. In certain places the contraction *w.r.t.* has been used for the phrase '*with respect to*.'

SECTION I

(Definition and birational transformation of the V-surface)

1. According to van der Waal, the pressure p, volume v and absolute temperature T of a given mass of gas conform to the well-known relation :

$$\left(p+\frac{a}{v^2}\right)(v-b)=\mathrm{R}T, \qquad \dots \quad (1)$$

where a, b, R are constants.

If we now take any three concurrent and orthogonal lines OX, OY, OZ as axes of co-ordinates, and take OX as the axis of pressure, OY as the axis of volume and OZ as the axis of (absolute) temperature, the three-dimensional graph of the equation (1) is evidently a surface of the fourth degree, whose Cartesian equation is

$$\left(x+\frac{a}{y^2}\right)(y-b)=\mathrm{R}z. \qquad \dots \quad (2)$$

This quartic surface (Ω) will be frequently designated as the V-surface. It may be remarked in passing that, when a, b are put $=0$ (as a first approximation), Ω degenerates into a paraboloid, of which the two systems of generating lines are parallel respectively to the planes $x=0$ and $y=0$. This trivial case will be ignored throughout this paper, so that the constants a, b (however insignificant) will be supposed to have *non-zero* values.

2. Elementary reasoning readily reveals the *ruled character* of the surface Ω, the *general* equations of the set of generating lines being

$$\left.\begin{aligned} y &= \lambda, \\ \text{and} \qquad x+\frac{a}{\lambda^2} &= \frac{\mathrm{R}}{\lambda-b}\,z, \end{aligned}\right\} \qquad \dots \quad (1)$$

where λ is a variable parameter.

To find the line of striction on Ω, we observe in the first place that, the generators being all parallel to the plane $y=0$, the shortest distance between any two consecutive members is parallel to the y-axis. If, then, (x', y', z') be the point, where an arbitrary generator, as defined by (1), is met by the shortest distance from the consecutive generator, *viz.*,

$$\left.\begin{aligned} y &= \lambda + d\lambda, \\ x + \frac{a}{(\lambda+d\lambda)^2} &= \frac{R}{\lambda + d\lambda - b} \cdot z, \end{aligned}\right\} \quad \ldots \quad (2)$$

the line $(x=x',\ z=z')$ must intersect the line (2). The condition for this to be possible is plainly

$$x' + \frac{a}{(\lambda+d\lambda)^2} = \frac{R}{\lambda+d\lambda-b} \cdot z'. \quad \ldots \quad (3)$$

Besides, we have

$$y'=\lambda \qquad \text{and} \qquad x'+\frac{a}{\lambda^2}=\frac{R}{\lambda-b}z'. \quad \ldots \quad (4)$$

Solving (3) and (4) for x', y', z' and omitting the dashes, we learn that the line of striction of Ω is a unicursal quartic curve, definable by the parametric equations :

$$x=\frac{a(\lambda-2b)}{\lambda^3}, \qquad y=\lambda, \qquad z=\frac{2a}{R}\cdot\frac{(\lambda-b)^2}{\lambda^3}.$$

3. We shall now establish a remarkable property of the surface Ω, *viz.*, that it admits of conversion into a plane by means of a *Cremona transformation* (*C.T.*).

To that end we observe firstly that the *C. T.*, definable by either of the two *equivalent* triads of equations :

$$\left.\begin{aligned} &(i) \qquad \xi=x+\frac{a}{y^2}, \qquad \eta=y-b, \quad \zeta=Rz, \\ &(ii) \qquad x=\xi-\frac{a}{(\eta+b)^2}, \quad y=\eta+b, \quad z=\frac{\zeta}{R}, \end{aligned}\right\} \quad \ldots \quad (I)$$

changes the surface Ω into the paraboloid

$$\xi\eta=\zeta\,.$$

Secondly this paraboloid can be turned into a plane (*viz.*, $X=Y$) by the *C. T.*, definable by either of the two equivalent sets of equations :

$$\left.\begin{aligned} &(iii) \qquad X=\frac{\zeta}{\xi}, \qquad Y=\eta, \quad Z=A\xi+B\eta+C\zeta\,; \\ &(iv) \qquad \xi=\frac{Z-BY}{A+CX}, \quad \eta=Y, \quad \zeta=\frac{X(Z-BY)}{A+CX}\,. \end{aligned}\right\} \quad \ldots \quad (II)$$

It follows conclusively that the *C. T.*, compounded of the two *C. T.*'s (I) and (II), converts the original surface Ω into a plane. The presence of the *arbitrary* constants A, B, C as well as the (obvious) *arbitrariness* in the selection of the two components $C.T.$'s bring home to one's mind that the surface Ω can be carried over into a plane by means of an infinitude of $C. T.$'s. Bearing in mind that transformability into a plane by aid of a $C. T.$ is a *characteristic* property of a unicursal surface, we infer that Ω is a unicursal surface. This can be substantiated more simply as follows.

Introduce a *rational*—but not necessarily integral—function of μ, say $\phi(\mu)$, and equate z to $\phi(\mu)$ in the equations (1) of Art. 2. Clearly, then, the surface Ω admits of the following *rational* parametric representation, *viz.*,

$$\left.\begin{aligned} x&=\frac{R\phi(\mu)}{\lambda-b}-\frac{a}{\lambda^2}, \\ y&=\lambda, \\ z&=\phi(\mu), \end{aligned}\right\} (\lambda,\ \mu \text{ being parameters}).$$

So once again the unicursal property of Ω is manifest. A third proof of the same result will be considered in the next article.

4. We shall now re-write the equation of Ω in the form :

$$(xy^2+a)\ (y-b)-Ry^2z=0, \qquad \ldots \quad (1)$$

and use the symbols L and M to denote respectively the two right lines, along which the plane at infinity is cut by the two planes :

$$x=0 \quad \text{and} \quad y=0.$$

Since xy^3 represents the only term of the *fourth* order in (1), we gather that the section of Ω by the plane at infinity is a (plane) quartic curve, consisting of the line M (counted thrice) and the line L (counted once). A cursory glance at the equations (1) of Art. 2 suggests that M is a *common transversal* (or director) of the ∞^1 of generators of the surface.

It is easy to see that M is a *triple line* of the surface Ω. For an *arbitrary* plane through M being taken in the form :

$$y=\lambda,$$

its complete curve of intersection with Ω plainly consists of the line M (counted thrice), and the line

$$y=\lambda, \quad x+\frac{a}{\lambda^2}=\frac{Rz}{\lambda-b}. \qquad \ldots \quad (2)$$

The inevitable conclusion is that M is a triple line, and that the line (2) lies wholly on Ω for all values of λ. This last result is already a proved fact (*cf.* Art. 2).

The surface Ω, endowed, as it is, with a triple line, must needs be *unicursal* and so each of its plane sections is a unicursal quartic. It must not be overlooked that this result is quite in consonance with the more general proposition which states that, if any algebraic surface of degree n possesses a multiple curve of degree $n-1$, this curve must be a right line and at the same time the surface must be unicursal.

SECTION II

(Systems of polar quadrics and the Hessian of Ω)

5. Let us now specify the position of an arbitrary point P by means of *homogeneous co-ordinates* (x, y, z, w), referred to the tetrahedron formed by the three (Cartesian) co-ordinate planes (yz), (zx), (xy) and the plane at infinity. Evidently, then, the first three homogeneous co-ordinates are the same as its Cartesian co-ordinates, whereas the fourth co-ordinate w may be put equal to unity. So the homogeneous equation of the surface Ω, as given by (1) of Art. 4, may be written in the symbolic form :

$$\phi(x, y, z, w)=0, \qquad \dots \quad (1)$$

where
$$\phi \equiv (xy^2+aw^3)(y-bw)-Ry^2zw.$$

Partial differentiations give

$$\left.\begin{aligned}
&\frac{\partial \phi}{\partial x}=y^3-by^2w\ ; \quad \frac{\partial \phi}{\partial y}=3xy^2-2bxyw-2Ryzw+aw^3\ ;\\
&\frac{\partial \phi}{\partial z}=-Ry^2w\ ; \quad \frac{\partial \phi}{\partial w}=-bxy^2-Ry^2z+3ayw^2-4abw^3\ ;\\
&\frac{\partial^2 \phi}{\partial x^2}=0\ ; \quad \frac{\partial^2 \phi}{\partial x \partial y}=3y^2-2byw\ ; \quad \frac{\partial^2 \phi}{\partial x \partial z}=0; \quad \frac{\partial^2 \phi}{\partial x \partial w}=-by^2\ ;\\
&\frac{\partial^2 \phi}{\partial y^2}=6xy-2bxw-2Rzw\ ; \quad \frac{\partial^2 \phi}{\partial y \partial z}=-2Ryw\ ;\\
&\frac{\partial^2 \phi}{\partial y \partial w}=-2\,bxy-2\,Ryz+3aw^2\ ; \quad \frac{\partial^2 \phi}{\partial z^2}=0\ ;\\
&\frac{\partial^2 \phi}{\partial z \partial w}=-Ry^2\ ; \quad \frac{\partial^2 \phi}{\partial w^2}=6ayw-12abw^2.
\end{aligned}\right\} \quad \dots \quad (1)$$

Palpably, then, the Hessian of Ω, which is geometrically definable as the locus of points whose polar quadrics are cones, and is analytically definable by the equation :

$$\begin{vmatrix} \frac{\partial^2\phi}{\partial x^2}, & \frac{\partial^2\phi}{\partial x\partial y}, & \frac{\partial^2\phi}{\partial x\partial z}, & \frac{\partial^2\phi}{\partial x\partial w} \\ \frac{\partial^2\phi}{\partial y\partial x}, & \frac{\partial^2\phi}{\partial y^2}, & \frac{\partial^2\phi}{\partial y\partial z}, & \frac{\partial^2\phi}{\partial y\partial w} \\ \frac{\partial^2\phi}{\partial z\partial x}, & \frac{\partial^2\phi}{\partial z\partial y}, & \frac{\partial^2\phi}{\partial z^2}, & \frac{\partial^2\phi}{\partial z\partial w} \\ \frac{\partial^2\phi}{\partial w\partial x}, & \frac{\partial^2\phi}{\partial w\partial y}, & \frac{\partial^2\phi}{\partial w\partial z}, & \frac{\partial^2\phi}{\partial w^2} \end{vmatrix} = 0,$$

takes the form :

$$\begin{vmatrix} 0, & \frac{\partial^2\phi}{\partial x\partial y}, & 0, & \frac{\partial^2\phi}{\partial x\partial w} \\ \frac{\partial^2\phi}{\partial y\partial x}, & \frac{\partial^2\phi}{\partial y^2}, & \frac{\partial^2\phi}{\partial y\partial z}, & \frac{\partial^2\phi}{\partial y\partial w} \\ 0, & \frac{\partial^2\phi}{\partial z\partial y}, & 0, & \frac{\partial^2\phi}{\partial z\partial w} \\ \frac{\partial^2\phi}{\partial w\partial x}, & \frac{\partial^2\phi}{\partial w\partial y}, & \frac{\partial^2\phi}{\partial w\partial z}, & \frac{\partial^2\phi}{\partial w^2} \end{vmatrix} = 0.$$

When expanded in terms of the first row and reduced, this equation can be easily put in the form :

$$\Delta^2 = 0,$$

where
$$\Delta \equiv \begin{vmatrix} \frac{\partial^2\phi}{\partial y\partial x}, & \frac{\partial^2\phi}{\partial y\partial z} \\ \frac{\partial^2\phi}{\partial w\partial x}, & \frac{\partial^2\phi}{\partial w\partial z} \end{vmatrix}.$$

Since $\Delta = -3Ry^4$ by (I), it follows that the Hessian of the V-surface is a *degenerate* surface of the eighth degree and consists simply of the plane $y=0$, counted *eight* times.

6. We know that the polar quadric of a point P (x, y, z) with respect to a surface, given by the Cartesian equation

$$F(x, y, z) = 0,$$

is a paraboloid, if, and only if, the determinant D, defined by

$$D \equiv \begin{vmatrix} \frac{\partial^2 F}{\partial x^2}, & \frac{\partial^2 F}{\partial x \partial y}, & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x}, & \frac{\partial^2 F}{\partial y^2}, & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x}, & \frac{\partial^2 F}{\partial z \partial y}, & \frac{\partial^2 F}{\partial z^2} \end{vmatrix}$$

vanishes at P. So in the general case (when $D \neq 0$ identically) the equation $D = 0$ represents the surface-locus of a point, whose polar quadric is a paraboloid. In the *exceptional* case when $D = 0$ *identically*, *every* polar quadric is a paraboloid.

When we apply the above lemma to the surface Ω, we set

$$F(x, y, z) = \phi(x, y, z, 1)$$

and note the three relations of I (Art. 5), *viz.*,

$$\frac{\partial^2 \phi}{\partial x^2} = 0, \quad \frac{\partial^2 \phi}{\partial z^2} = 0 \text{ and } \frac{\partial^2 \phi}{\partial x \partial z} = 0.$$

Manifestly then $D = 0$ independently of x, y, z. We cannot therefore escape the conclusion that *the polar quadric of every point with respect to the V-surface is a paraboloid.* This result is susceptible of independent verification as follows.

The Cartesian equation of the polar quadric of a point P (x', y', z') (with respect to Ω) is

$$\left(x\frac{\partial}{\partial x'} + y\frac{\partial}{\partial y'} + z\frac{\partial}{\partial z'} + w\frac{\partial}{\partial w'} \right)^2 \cdot \phi(x', y', z', w') = 0,$$

where w, w' are to be put $= 1$ after differentiations. When the left side is expanded and the relations (I) of Art. 5 are utilised, the equation can be easily thrown into the symbolic form :

$$y(\lambda x + \mu y + \nu z) + lx + my + nz + p = 0, \qquad \dots (1)$$

where
$$\left.\begin{aligned} \lambda &\equiv y'(3y' - 2b) \; ; \quad \mu \equiv 3x'y' - bx' - Rz' \; ; \\ \nu &\equiv -2Ry' \; ; \quad l = -by'^2 \; ; \\ m &\equiv -2Ry'z' - 2bx'y' + 3a \; ; \quad n = -Ry'^2 ; \\ p &\equiv 3a(y' - 2b). \end{aligned}\right\} \qquad \dots (2)$$

Inasmuch as the quadratic terms are the product of the two linear factors :

$$y \text{ and } \lambda x + \mu y + \nu z,$$

we conclude that, wherever the point P may lie, its polar quadric with respect to the surface Ω is a paraboloid, one system of whose generating lines are parallel to the *fixed* plane $y = 0$.

The geometrical explanation is not far to seek. For a multiple curve (of multiplicity p), known to lie on a surface II must be a multiple curve (of multiplicity $p-q$) on the qth polar surface of every point (*w.r.t.* II), provided that $q < p$. Applying this lemma to the surface Ω, and recollecting (Art. 4) that M is a triple line on Ω, we infer that the polar quadric of an *arbitrary* point (*w.r.t.* Ω) must have M for a multiple curve of multiplicity $1 (\equiv 3-2)$. In other words, every polar quadric (of Ω) must have M for an ordinary generator and must $\therefore$ cut the plane at ∞ along two right lines, one of which is M. That is to say, an arbitrary polar quadric of Ω has a *degenerate 'conic at infinity'*, and is accordingly a paraboloid. This corroborates the previous result.

7. Let us now look for the locus of points, whose polar quadrics *w.r.t.* Ω are cylinders. Since a cylinder is the only type of quadric, which is at once a paraboloid and a cone, we promptly perceive that the necessary and sufficient condition for the (paraboloidal) polar quadric of a point P (x', y', z') to be a cylinder is that P should lie on the Hessian $(y^8 = 0)$. Thus any point, whose polar quadric is a cylinder, may be taken as $(x', 0, z')$, (where x', z' are *arbitrary*), and the actual equation of the associated (cylindrical) polar quadric is by (1) of Art. 6 seen to be

$$\mu y^2 + my + p = 0,$$

where $\mu = -bx' - Rz'$, $m = 3a$ and $p = -6ab$. Obviously the cylinder is of the degenerate type and consists merely of two parallel planes.

We may summarise our conclusions in the following manner :—

Whereas all possible polar quadrics—numbering, of course, ∞^3—are paraboloids, only a ∞^2 of them are cylinders, and consist simply of pairs of parallel planes. Furthermore the ∞^2 of points, whose polar quadrics are cylindrical, are all situated on the Hessian.

8. Before we close this chapter we shall touch briefly on the system of polar cubics of the surface Ω.

In accordance with the geometrical lemma (quoted in the previous article), it follows that the triple line of Ω, *viz.*, the line M, is a double line on *every* polar cubic. Remembering that a cubic surface, endowed with a double line, is either a cubic scroll or else a cubic cone, we gather that every polar cubic of Ω is a ruled surface. That any such polar cubic is a scroll (and *not* a cone) is obvious from the fact that its double line M is situated wholly in the plane at infinity. Alternative reasoning also points to the same conclusion.

For the homogeneous equation to the polar cubic of (x', y', z', w') is

$$x' \frac{\partial \phi}{\partial x} + y' \frac{\partial \phi}{\partial y} + z' \frac{\partial \phi}{\partial z} + w' \frac{\partial \phi}{\partial w} = 0.$$

Putting $w, w' = 1$ and using (I) of Art. 5, we easily derive its Cartesian equation in the form :

$$x'(y^3 - by^2) + y'(3xy^2 - 2bxy - 2Ryz + a) - Rz'y^2$$
$$- bxy^2 - Ry^2z + 3ay - 4ab = 0$$

Manifestly, then, this is a scroll, the series of generators being given by

$$y = \mu \text{ and } x(3y'\mu^2 - b\mu^2 - 2by'\mu) - Rz(2\mu y' + \mu^2)$$
$$+ x'(\mu^3 - b\mu^2) + ay' - R\mu^2 z' + 3a\mu - 4ab = 0,$$

where μ is a parameter. This confirms the previous result.

SECTION III

(*Location of the critical point of* Ω)

9. The aggregate of tangents to any given surface Π evidently forms a *line-complex*, which includes within its fold the *inflexional congruence* Σ, made up of the ∞^2 of inflexional tangents (to Π). Clearly the *order* of this congruence Σ is = the no. of lines that pass through an arbitrarily *assigned* point P and is therefore equal to no. of lines that are parallel to an arbitrarily *assigned* line L. For special positions of the point P or of the line L, this number may suffer a diminution and so may fall short of the *order*.

The avowed object of the present section is to find a point P (α, β, γ), lying on the surface Ω and having one of the two inflexional tangents (thereat) parallel to the y-axis. So one of the two inflexional tangents to Ω at P must be the line (N)

$$x = \alpha,\ z = \gamma. \qquad \dots \quad (1)$$

Making the substitutions (1) in the equation of Ω, *viz.*,

$$(xy^2 + a)(y - b) - Ry^2z = 0, \qquad \dots \quad (2)$$

we readily perceive that the resulting equation in y, *viz.*,

$$(\alpha y^2 + a)(y - b) - R\gamma y^2 = 0$$

must be identical with

$$(y - \beta)^3 = 0.$$

Accordingly the requisite conditions—at once necessary and sufficient—are

$$3\beta = \frac{b\alpha + R\gamma}{\alpha}, \quad 3\beta^2 = \frac{a}{\alpha}, \quad \beta^3 = \frac{ab}{\alpha}.$$

These lead to

$$\alpha = \frac{a}{27b^2},\ \beta = 3b,\ \gamma = \frac{8}{27}\cdot\frac{a}{Rb}. \qquad \dots \quad (\mathrm{I})$$

It can be easily seen that the values of α, β, γ, as given by (I), are respectively the *critical pressure*, *critical volume* and *critical temperature* (of the given

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mass of gas). If we now introduce the nomenclature *critical point* to represent that particular point (on Ω), whose Cartesian coordinates (taken in order) are the critical press., critical vol., and critical temp., we are entitled to present our conclusions in the following manner :—

The critical point is geometrically designable as the uniquely determinate point (on Ω), one of whose inflexional tangents is parallel to the axis of volume.

10. In order to find the second inflexional tangent (N') at $P(\alpha, \beta, \gamma)$, we may take its equations in the form :

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \; (= r), \tag{1}$$

where $l : m : n$ are quantities to be determined. Putting

$$x, y, z = \alpha + lr, \; \beta + mr, \; \gamma + nr,$$

in the equation of Ω, *viz.* (2) of Art. 9, we find that the four points of intersection of (1) with Ω depend on the following biquadratic in r :—

$$Ar^4 + Br^3 + Cr^2 + Dr = 0, \tag{2}$$

where
$$\left.\begin{aligned} A &\equiv lm^3 ; \quad B \equiv 3lm^2\beta + m^3\alpha - m^2(bl + Rn) ; \\ C &\equiv 3lm\beta^2 + 3m^2\alpha\beta - m^2(b\alpha + R\gamma) - 2m\beta(bl + Rn) ; \\ D &\equiv l\beta^3 + 3m\alpha\beta^2 - \beta^2(bl + Rn) - 2m\beta(b\alpha + R\gamma) + am. \end{aligned}\right\} \tag{3}$$

Plainly the conditions (necessary as well as sufficient) for (1) to be an inflexional tangent are

$$C = 0 \text{ and } D = 0.$$

When the values of α, β, γ as given by (I) of Art. 9 are made use of, the two relations last written can be thrown into the forms :

$$\left.\begin{aligned} m(7bl - 2Rn) &= 0, \\ \text{and} \qquad 2bl - Rn &= 0. \end{aligned}\right\}$$

So the two sets of values of $l : m : n$ are

$$\left.\begin{aligned} &0 : 1 : 0 ; && (4) \\ \text{and} \qquad &R : 0 : 2b. && (5) \end{aligned}\right\}$$

For obvious reasons the first solution refers to the inflexional tangent N considered in Art 9 As a matter of course, the second solution must then refer to the other inflexional tangent N', whose equations may be written as

$$\frac{x-\alpha}{R} = \frac{y-\beta}{0} = \frac{z-\gamma}{2b}.$$

A comparison of (4) and (5) reveals the mutual perpendicularity of the two inflexional tangents N and N' at the critical point P. The irresistible conclusion is that *the indicatrix of the V-surface Ω at this point is an equilateral hyperbola and that the mean curvature (of the surface) vanishes thereat.*

The tangent plane to the surface Ω at P, containing, as it does, the two inflexional tangents N, N', must then have for its Cartesian equation

$$2b(x-\alpha)-R(z-\gamma) = 0,$$

i.e.,

$$18b^2x-9Rbz+2a = 0. \qquad \ldots \quad (6)$$

This certainly admits of independent verification.

11. When we look for the polar quadric of P *w. r. t.* the surface Ω, we have to take recourse to (1) of Art 6 and to write

$$x' = \alpha = \frac{a}{27b^2}\ ; \quad y' = \beta = 3b\ ; \quad z' = \gamma = \frac{8}{27}\cdot\frac{a}{Rb} \quad \text{(Art. 9)}$$

in (2) of Art. 6. So the polar quadric (Γ) of P may be exhibited in the form :

$$y(\lambda x+\mu y+\nu z)+lx+my+nz+p = 0, \qquad \ldots \quad (1)$$

where

$$\left.\begin{aligned} &\lambda = 21b^2\ ; \quad \mu = 0\ ; \quad \nu = -6Rb\ ; \quad l = -9b^3\ ; \\ &m = a\ ; \quad n = -9Rb^2; \quad p = 3ab. \end{aligned}\right\}$$

The equation (1) of Γ being re-written in the form :

$$3by(7bx-2Rz)-9b^3x+ay-9Rb^2z+3ab = 0,$$

it is not difficult to see that Γ contains the whole length of each of the two inflexional tangents (N, N') at P. Having regard to the obvious fact that Γ touches the tangent plane to Ω at P as given by (6) of the preceding article, we may re-state the set of results in the following garb :—

The polar quadric Γ of the critical point P with respect to the surface Ω is a hyperbolic paraboloid, touching Ω at P. Further, the two inflexional tangents of Ω, that pass through P, are none other than the two generators of Γ, that pass through the very same point. That is to say, Γ and Ω possess the *same* tangent plane and the *same* pair of inflexional tangents at the point P.

SECTION IV

Certain organic curves on the V-surface

12. For the surface Ω :

$$\left(x+\frac{a}{y^2}\right)(y-b) = Rz,$$

the partial differential coefficients

$$\frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y}, \ \frac{\partial^2 z}{\partial x^2}, \ \frac{\partial^2 z}{\partial x \partial y}, \ \frac{\partial^2 z}{\partial y^2},$$

(symbolised respectively as p, q, r, s, t) are given by

$$\left.\begin{aligned} Rp &= y-b, \quad Rq = x-\frac{a}{y^3}(y-2b), \\ Rr &= 0, \; Rs = 1, \text{ and } Rt = \frac{2a}{y^4}(y-3b). \end{aligned}\right\}$$

So the mean curvature H and the specific (or Gaussian) curvature K at a point on Ω are easily found to be

$$\left.\begin{aligned} H &= \frac{(1+p^2)t-2pqs+(1+q^2)r}{(1+p^2+q^2)^{\frac{3}{2}}} \\ &= \frac{\frac{2a}{y^4}(y-3b)\{(y-b)^2+R^2\}-2(y-b)\left\{x-\frac{a}{y^3}(y-2b)\right\}}{\left[R^2+(y-b)^2+\left\{x-\frac{a}{y^3}(y-2b)\right\}^2\right]^{\frac{3}{2}}}, \\ \text{and} \quad K &= \frac{rt-s^2}{(1+p^2+q^2)^2} = -\frac{R^2}{\left[R^2+(y-b)^2+\left\{x-\frac{a}{y^3}(y-2b)\right\}^2\right]^2}. \end{aligned}\right\} \quad \ldots(\mathrm{I})$$

Since K is negative (except when $y = 0$), it appears that the surface Ω is *anticlastic* throughout the finite portion of space.

13. Evidently the curves of *constant* mean curvature (on Ω) are to be found by equating to a constant the value of H, as given by (I) of the foregoing article. In particular, the curve of *zero* mean curvature $(H = 0)$ is the intersection of Ω with the sextic surface :

$$a(y-3b)\{(y-b)^2+R^2\}-y(y-b)\{xy^3-a(y-2b)\} = 0.$$

It is a pleasant exercise to verify that this equation is satisfied by the co-ordinates (α, β, γ) of the critical point P (Art. 9). The immediate inference is that the curve of no mean curvature goes through P. This is however, a foregone conclusion, seeing that the locus of points of zero mean curvature (on *any* surface) is essentially the same as the locus of points whose inflexional tangents are orthogonal, or as the locus of points whose indicatrices are rectangular hyperbolas (Art. 10).

The specific curvature at the point P being given by

$$K = -\frac{R^2}{(R^2+4b^2)^2},$$

the two principal radii of curvature at the same point must be

$$\pm \frac{R^2+4b^2}{R}.$$

In like manner a curve of constant *non-zero* Gaussian curvature (say, $-c^2$) is the intersection of Ω with the surface :

$$(y-b)^2+\left\{x-\frac{a}{y^3}(y-2b)^2\right\}=\pm\frac{R}{c}-R^2.$$

Manifestly the parabolic curve—or what is the same thing, the curve of *zero* specific curvature—is *non-existent* on the surface Ω.

REFERENCES

[1] Salmon's *Geometry of Three Dimensions.*
[2] Basset's *Treatise on Surfaces.*
[3] Hudson's *Cremona Transformations.*
[4] Bagchi's *Geometrical Analysis.*